Obtaining Total Motion from Spectral Plots

by George M. Hieber

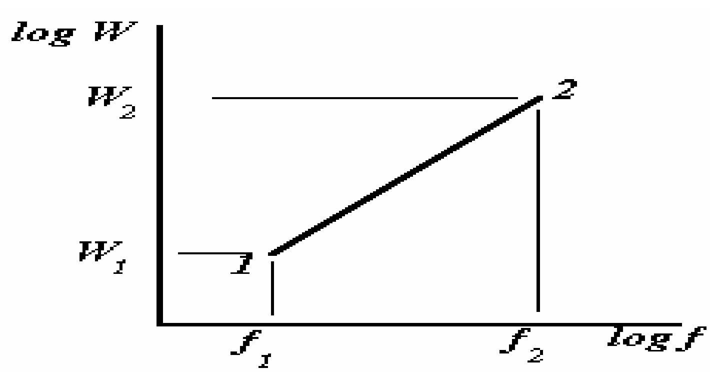
Given a spectral density plot of G^2/Hz, it is sometimes necessary to determine the total G_{rms} of the motion, which includes the motion contributed from all frequency bands. In order to do this, it is first necessary to add the spectral density in all bands throughout the spectrum to obtain the total G^2. The square root of G^2 is then taken to get the required G_{rms}.

In a region of the spectrum where the spectral density is uniform (where it plots as a horizontal straight line), the total G^2 is obtained merely by multiplying the spectral density in that region by the bandwidth of the region: $G^2 = \left(G^2/Hz\right)\left(f_2 - f_1\right)$. Essentially, this is simply calculating the area under the line.

Frequently, there are parts of the spectrum where the spectral density is either rising or falling with frequency. This results in plots which have sloped straight lines. These lines have slopes which are multiples of 3dB per octave, as the dynamic response of physical systems tends to become asymptotic to such lines. The severity of the slope is dependent upon the number of energy storage elements (masses and/or springs) which are active at a particular corner frequency or natural frequency.

The plots of spectral density versus frequency are log-log plots. If the same data which appears as straight line on a log-log grid is plotted on a linear grid, these data would appear as a curve. Because of this, it is not correct to try to get the area under a sloped line on a log-log plot by assuming the area consists of a triangle on a rectangular pedestal, even though it appears that way. Special formulas must be used. Derivations follow.

Assume a portion of a spectral plot as shown:



Where W is power spectral density (PSD) in $(engineering\ units)^2 / Hz$.

(In vibration parlance, PSD is usually in G^2/Hz)

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From the formula for a straight line: $y = mx + b$

Substituting from the graph,

$$\log W_1 = m \log f_1 + b$$

$$\log W_2 = m \log f_2 + b$$

Subtracting,

$$\log W_2 - \log W_1 = m[\log f_2 - \log f_1]$$

Or $$\log (W_2/W_1) = m[\log(f_2/f_1)]$$

Taking antilogs, $W_2/W_1 = (f_2/f_1)^m$

With power phenomena, such as PSD, a change in intensity of 3dB is a change in power of two to one. With frequency, an octave is a change of frequency of two to one. So, for example, if the frequencies in the above equation are one octave apart, $(f_2/f_1) = 2$. Then, if the power is changed by a factor of two (3dB change) per octave, the exponent *m* must equal 1. If the power were to change by a factor of four per octave, the exponent must equal 2. For a power change of eight, the exponent must be 3. And so forth. This means that the exponent *m* can be expressed as *R/3,* where *R* is the slope of the line in dB per octave.

Therefore, $$\frac{W_2}{W_1} = \left(\frac{f_2}{f_1}\right)^{\frac{R}{3}}$$

Knowing any four values, the remaining unknown can be found.

To solve for the total G^2 under the sloped line, the values under the line must be integrated:

$$G^2 = \int_{f_1}^{f_2} W df$$

Substituting $W = W_1\left(\frac{f}{f_1}\right)^{\frac{R}{3}}$

$$G^2 = W_1 \int_{f_1}^{f_2} \left(\frac{f}{f_1}\right)^{\frac{R}{3}} df = W_1 f_1 \int_{f_1}^{f_2} \left(\frac{f}{f_1}\right)^{\frac{R}{3}} d\left(\tfrac{f}{f_1}\right)$$

$$G^2 = \frac{W_1 f_1}{\frac{R}{3}+1}\left[\left(\tfrac{f_2}{f_1}\right)^{\frac{R}{3}+1} - 1\right]$$

Finally, $$G^2 = \frac{3W_1 f_1}{R+3}\left[\left(\frac{f_2}{f_1}\right)^{\frac{R+3}{3}} - 1\right] \qquad (1)$$

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If motion in terms of *velocity* is desired, V_{rms} is the quantity of interest, where V is in in/sec. Since the *PSD* is normally scaled in units of G^2/Hz, which is dimensioned as 1/seconds (The expression G itself is non-dimensional), the *PSD* must first be changed to units of acceleration squared, which is $\frac{in^2}{\sec^4}$ per Hz. So as velocity equals acceleration $\div w$,

$$V^2 = \frac{accel^2}{\omega^2} = \frac{(gG)^2}{\omega^2} = \frac{(386)^2 G^2}{\omega^2} = \frac{3774}{f^2} G^2 = \frac{3774}{f^2} \int_{f_1}^{f_2} W df$$

A mathematical expression which changes as $1/f$ with increasing frequency will graph as a sloping line at –3dB per octave, and if it changes as $1/f^2$ with increasing frequency it will graph as a sloping line at –6dB per octave. This applies to acceleration to velocity conversion: Compared to acceleration, velocity drops off as frequency increases, equal to acceleration times $1/f$, and velocity squared is equal to acceleration times $1/f^2$; therefore, an additional –6dB per octave must be taken into account when plotting V^2 compared to plotting G^2:

To accommodate this, the expression $W = W_1 \left(\frac{f}{f_1}\right)^{\frac{R}{3}}$ becomes $V^2 = \frac{3774\, W_1}{f_1^2} \left(\frac{f}{f_1}\right)^{\frac{R-6}{3}}$

Then $V^2 = \frac{3774}{f_1^2} \int_{f_1}^{f_2} W df = \frac{3774\, W_1}{f_1^2} \int_{f_1}^{f_2} \left(\frac{f}{f_1}\right)^{\frac{R-6}{3}} df = \frac{3774\, W_1 f_1}{f_1^2} \int_{f_1}^{f_2} \left(\frac{f}{f_1}\right)^{\frac{R-6}{3}} d\left(\frac{f}{f_1}\right)$

$$V^2 = \frac{3774\, W_1}{f_1 \left(\frac{R-6}{3} + 1\right)} \left[\left(\frac{f_2}{f_1}\right)^{\frac{R-6}{3}+1} - 1 \right]; \qquad Finally,\ V^2 = \frac{11322\, W_1}{(R-3)\, f_1} \left[\left(\frac{f_2}{f_1}\right)^{\frac{R-3}{3}} - 1 \right] \qquad (2)$$

If motion in terms of *displacement* is desired, Y_{rms} is the quantity of interest, where Y is in inches. In this case, since displacement equals *acceleration* $\div\, \omega^2$,

$$Y^2 = \frac{(gG)^2}{\omega^4} = \frac{(386\ G)^2}{(2\pi f)^4} = \frac{95.6}{f^4} G^2$$

Since $1/f^4$ is a change of –12dB per octave, $Y^2 = \frac{95.6\ W_1}{f^4} \left(\frac{f}{f_1}\right)^{\frac{R-12}{3}}$

$$Y^2 = \frac{95.6}{f^4} \int_{f_1}^{f_2} W df = \frac{95.6}{f_1^4} W_1 f_1 \int_{f_1}^{f_2} \left(\frac{f}{f_1}\right)^{\frac{R-12}{3}} d\left(\frac{f}{f_1}\right)$$

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$$Y^2 = \frac{95.6\ W_1}{f_1^3\left(\frac{R-12}{3}+1\right)}\left[\left(\frac{f_2}{f_1}\right)^{\frac{R-12}{3}+1} - 1\right] \quad ; \qquad Finally,\quad Y^2 = \frac{286.8\ W_1}{f_1^3\,(R-9)}\left[\left(\frac{f_2}{f_1}\right)^{\frac{R-9}{3}} - 1\right] \quad (3)$$

The above equations cannot be used for computing G^2 when R = –3, for V^2 when R = +3, and for Y^2 when R = +9, as these values of slope cause the applicable equation to "blow up." In these cases, special equations must be used.

Starting with the ratio equations,

If R= –3: $$W_2 = W_1\left(\frac{f_2}{f_1}\right)^{\frac{-3}{3}} = W_1\left(\frac{f_2}{f_1}\right)^{-1} = W_1\left(\frac{f_1}{f_2}\right)$$

Similarly, if R= +3: $$V_2 = V_1\left(\frac{f_2}{f_1}\right)^{\frac{3-6}{3}} = V_1\left(\frac{f_1}{f_2}\right)$$

And if R= +9: $$Y_2 = Y_1\left(\frac{f_2}{f_1}\right)^{\frac{9-12}{3}} = Y_1\left(\frac{f_1}{f_2}\right)$$

For G^2 when R= –3:

$$G^2 = W_1 f_1 \int_{f_1}^{f_2} \frac{df}{f} = W_1 f_1\left[ln\ f_2 - ln\ f_1\right] = W_1 f_1\, ln\left(\frac{f_2}{f_1}\right)$$

or $$G^2 = 2.3\ W_1 f_1\, log_{10}\left(\frac{f_2}{f_1}\right) \qquad (4)$$

In a similar fashion, for V^2 when R= +3:

$$V^2 = \frac{3774\ W_1 f_1}{f_1^2}\int_{f_1}^{f_2}\frac{df}{f} = \frac{3774\ W_1}{f_1}\, ln\left(\frac{f_2}{f_1}\right)$$

or $$V^2 = \frac{8680\ W_1}{f_1}\, log_{10}\left(\frac{f_2}{f_1}\right) \qquad (5)$$

And for Y^2 when R= +9:

$$Y^2 = \frac{95.6\ W_1 f_1}{f_1^3}\int_{f_2}^{f_2}\frac{df}{f} = \frac{95.6\ W_1}{f_1^2}\, ln\left(\frac{f_2}{f_1}\right)$$

or $$Y^2 = \frac{220\ W_1}{f_1^2}\, log_{10}\left(\frac{f_2}{f_1}\right) \qquad (6)$$

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When using these equations to determine whether an exciter is capable of generating required overall G_{rms} or V_{rms} values, either equations (1) and (4) or (2) and (5) are used, as applicable, to solve for the total squared quantities in each of any defined frequency bands. After summing these squared values, the square root is extracted from the sum to get the rms value. Finally, the rms value is multiplied by 3 to account for the three sigma peaks which must be accommodated.

If the displacement capability of the exciter is of interest, then Y_{rms} is calculated by the use of equations (3) and (6). After summing, extracting the square root and multiplying by 3 as described in the previous paragraph, the result must then be multiplied by 2 to take into account the fact that peak to peak motion must be accommodated in the distance between armature stops.

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